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A NETWORK SOLUTION OF THE 1-DIMENSIONAL INVERSE PROBLEM BY TIME--ETC(U)
JAN 82 D C YOULA, G GNAVI F30602-78-C-0048
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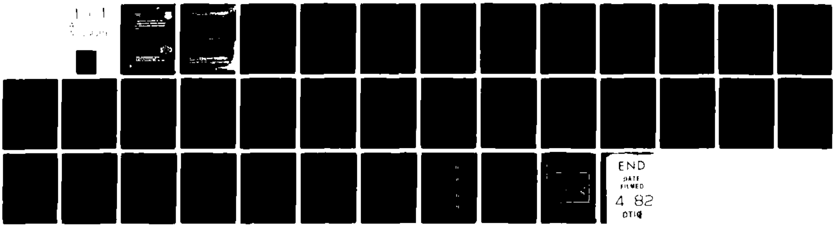

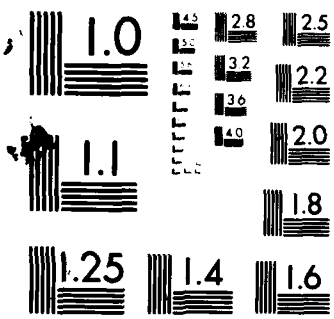

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REPORT DOCUMENTATION PAGE

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1. REPORT NUMBER: RADC-TR-81-381
2. GOVT ACCESSION NO.: AD-A112 629
3. RECIPIENT'S CATALOG NUMBER:
4. TITLE (and Subtitle): A NETWORK SOLUTION OF THE 1-DIMENSIONAL INVERSE PROBLEM BY TIME-DOMAIN REFLECTOMETRY, PART 1
5. TYPE OF REPORT & PERIOD COVERED: Phase Report
6. PERFORMING ORG. REPORT NUMBER: N/A
7. AUTHOR(s): D.C. Youla, G. Gnavi
8. CONTRACT OR GRANT NUMBER(s): F30602-78-C-0048
9. PERFORMING ORGANIZATION NAME AND ADDRESS: Polytechnic Institute of New York, Farmingdale NY 11735
10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS: 01727802
11. CONTROLLING OFFICE NAME AND ADDRESS: Rome Air Development Center (COTD), Griffiss AFB NY 13441
12. REPORT DATE: January 1982
13. NUMBER OF PAGES: 38
14. MONITORING AGENCY NAME & ADDRESS (if different from Controlling Office): Same
15. SECURITY CLASS. (of this report): UNCLASSIFIED
15a. DECLASSIFICATION/DOWNGRADING SCHEDULE: N/A
16. DISTRIBUTION STATEMENT (of this Report): Approved for public release; distribution unlimited
17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report): Same
18. SUPPLEMENTARY NOTES: RADC Project Engineer: Haywood E. Webb (COTD)
19. KEY WORDS (Continue on reverse side if necessary and identify by block number):
NDN Uniform Transmission Line
Identification
Circuit Theory
20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

This Part 1 of a projected 2-part study initiates an indepth network study of the inverse problem for 1-dimensional passive, lossless smooth dispersive nonuniform structures. The diagnostic procedure is via time-domain pulse-reflectometry and rests on a new continuous-discrete inversion formula.

It is expected that this formula, when used together with the weak convergence principle described in Section V, will lead to a well-posed (over)

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solution involving a coupled pair of Fredholm integral equations of the second kind.

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I. INTRODUCTION

It has only recently been realized in engineering circles, largely due to the outstanding pioneering work of Gopinath and Sondhi [5, 6], that the solution of the inverse problem for the classical lossless nonuniform line possesses a simple and natural time-domain setting. This simplicity is directly attributable to the fact that the expression of causality in time is incomparably less complicated than its expression in terms of frequency behavior.

Nevertheless, the problem of diagnostics for dispersive passive, lossless, smooth 1-dimensional tapers remains open despite the fact that its discrete counterpart has already been solved [2, 3]. In this part I of a projected 2-part study we demonstrate that the results obtained in Ref. 3 yield the deep continuous-discrete inversion formula (47) which provides a solid network foundation for our entire approach. In fact, when coupled with the weak convergence principle described in Section V, it immediately leads to an effective integral equation for the logarithmic taper P(x) of the telegrapher's line (theorem 2) in terms of the input impulse response.

We should like to stress that our conceptualization of a smooth taper is based squarely on physical considerations and the role of dispersion is given a quantitative characterization.

It is also expected that the final solution will involve a pair of coupled Fredholm integral equations of the second kind. Thus, the inverse problem, in its full generality, appears to possess the essential property of well-posedness. Moreover, the corresponding numerical algorithms are of the Levinson type and the theory of orthogonal polynomials on the unit circle can be exploited to advantage.

As regards notation, most of it is self-explanatory. However, we point out that for a matrix A, $\bar{A}$, A', $A^*(\equiv\bar{A}')$ and det A denote its complex conjugate, transpose, adjoint and determinant, respectively. As usual, column-vectors are written as $\underset{\sim}{a}$, $\underset{\sim}{b}$ or as $\underset{\sim}{x} = (x_1, x_2, \ldots, x_n)'$ if it is desirable to exhibit its components, and $\underset{\sim}{0}_n$, 1_n, $0_{m,n}$ stand for the n-dimensional zero vector, the nxn identity matrix and the mxn zero matrix, respectively. We also write $A \geq 0_n$ to indicate that the hermitean

matrix $A = A^*$ is nonnegative-definite. Lastly, if A(p) is any real meromorphic matrix function of p, it is convenient for typographical reasons to let

$$A_*(p) \equiv A'(-p) \quad .$$

Clearly, if $p = j\omega$, ω real, $A_*(j\omega) = A^*(j\omega)$.

Since this is a research report our demands on the reader's knowledge vary widely. Nevertheless, a solid exposure to the positive-real concept and its applications to insertion-loss design [1] should suffice for the most part.

II. THE MODEL

From an engineering point of view it appears reasonable to conceptualize a continuous 1-dimensional electrical taper (Fig. 1a) as the "physical" limit of a discrete cascade of generic type shown in Fig. 1b. If the taper is also internally passive and lossless and operates in a single-mode environment, it is then natural to impose the following constraints.

C_1) Every one of the n LLN's is a lumped passive, lossless 2-port.

C_2) The n+1 lines are all ideal TEM and possess the same 1-way delay $\tau > 0$ but arbitrary positive characteristic impedances $R_0, R_1, \ldots, R_n$. In our opinion, the presence of the initial line (of characteristic impedance R_0) is indispensable and serves to provide the necessary single-mode transition between the actual input and the actual start of the taper. (In any case, only experiment can decide for or against this assumption.)

C_3) The total 1-way delay

$$T = (n+1)\tau \tag{1}$$

remains constant as $n \to \infty$.

Let[1] $z_m(p)$ denote the impedance seen looking into the input side of LLN m with its output closed on R_m (Fig. 2a). By definition,

$$s_m(p) = \frac{z_m(p) - R_{m-1}}{z_m(p) + R_{m-1}} \quad , \quad m = 1 \to n \quad , \tag{2}$$

is the corresponding normalized "junction" reflection coefficient. Given that the load termination is passive but unknown, the purpose of internal diagnostics for the discrete cascade is to determine as much as possible about these coefficients by means of measurements carried out exclusively from the input side of the structure.

Now if the notion of a physical limit is meaningful, LLN m and the line of characteristic impedance R_m contiguous on its right must constitute an accurate representation of the properties of a section of continuous taper situated between $x=m\tau$ and $x=(m+1)\tau$, at least in the limit of small τ.[2] We

[1] $p = \sigma + j\omega$ is the complex-frequency variable.

[2] The analogy with the discrete model requires that x should equal the time in secs. for an electromagnetic pulse injected at x=0 to reach the cross-section identified by the abscissa x.

are therefore led to introduce the following interpretive assumptions.

A_1) There exists a positive and continuously differentiable function R(x) which plays the role of a "local" characteristic impedance for the continuous taper.

A_2) Let $z(x, p; \tau)$ denote the impedance seen looking into a section of continuous taper lying between x and $x+\tau$ which at $x+\tau$ is closed on the resistance $R(x+\tau)$ (Fig. 2b). Then, there exists a function $\eta(x, p)$ which is continuously differentiable in x for every fixed p in Re p > 0, and real and rational in p for every fixed x such that[3]

$$\frac{z(x, p; \tau)}{R(x+\tau)} = 1 + 2\tau\eta(x, p) + o(x, p; \tau) \quad , \tag{3}$$

$$0 \leq x < T \quad .$$

A_3) For $m\tau \to x > 0$, x fixed, and for p fixed in Re p > 0,

$$\frac{z(x, p; \tau) - R(x)}{z(x, p; \tau) + R(x)} = s_m(p) + o(x, p; \tau) \quad . \tag{4}$$

As an immediate consequence of the above three assumptions it is easily shown that as $m\tau \to x > 0$, x fixed,

$$\lim_{\tau \downarrow 0} \frac{s_m(p)}{\tau} = P(x) + \eta(x, p) \tag{5}$$

where

$$P(x) = \frac{1}{2} \cdot \frac{d \ln R(x)}{dx} \tag{6}$$

is the so-called logarithmic taper. The two functions P(x) and $\eta(x, p)$ delimit the generic taper profile while formula (5) supplies the means for their calculation through the discrete approximation.

Our first objective is to derive an expression for $s_m(p)$ in terms of input data. Since interaction with the load must be avoided it is natural to seek to develop a procedure based on time-domain pulse-reflectometry.

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[3] $\lim_{\tau \downarrow 0} \frac{o(x, p; \tau)}{\tau} = 0$, uniformly in x.

III. DIAGNOSTICS FOR THE DISCRETE CASCADE

Let the discrete cascade be excited through the input current driver $i(t) = \delta(t)$ and let $h^{(n)}(t)$ denote the corresponding input voltage impulse response $v(t)$ under initially quiescent conditions.

Evidently, because the round-trip travel time for each line is 2τ secs., $h^{(n)}(t)$ is independent of the load for all t in the interval $0 \leq t < 2(n+1)\tau$. In addition, in this same time interval $h^{(n)}(t)$ admits a decomposition into a sum of "reflected" waves of the form,

$$\frac{h^{(n)}(t)}{R_o} = \delta(t) + \sum_{m=1}^{n} 2h_m(t-2m\tau) \tag{7}$$

whose properties are summarized in theorem 1.

Theorem 1 (Appendix). Let

$$h_m(t) \Leftrightarrow c_m(j\omega) = \int_{-\infty}^{\infty} e^{-j\omega t} h_m(t)\, dt \quad , \tag{8}$$

$$m = 1 \to n \quad ,$$

and let

$$\Delta_m(p) = \det \begin{bmatrix} 1 & c_{1*}(p) & c_{2*}(p) & . & . & . & c_{m*}(p) \\ c_1(p) & 1 & c_{1*}(p) & . & . & . & c_{m-1*}(p) \\ c_2(p) & c_1(p) & 1 & . & . & . & c_{m-2*}(p) \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ c_m(p) & c_{m-1}(p) & c_{m-2}(p) & . & . & . & 1 \end{bmatrix} , \tag{9}$$

$$m = 0 \to n \quad .$$

Then,

1) every Laplace transform $c_m(p)$ is a real rational function of p analytic in $\mathrm{Re}\, p \geq 0$, $p = \infty$ included. (Thus, $h_m(t)$ vanishes for $t < 0$, $m = 1 \to n$.)

2) With the possible exception of a finite number of ω's at most,

$$\Delta_m(j\omega) > 0 \quad , \quad m = 0 \to n \quad , \tag{10}$$

ω real.

3) Let

$$S^{(m)}(p) = \left[\begin{array}{c|c} s_m(p) & s_{12}^{(m)}(p) \\ \hline s_{21}^{(m)}(p) & s_{22}^{(m)}(p) \end{array}\right] \tag{11}$$

denote the scattering matrix of LLN m normalized to R_{m-1} on the left and R_m on the right, let $d_m(p) = \det S^{(m)}(p)$ and let $\Delta_m'(p)$ denote the minor of $\Delta_m(p)$ that results by suppressing its first row and last column. Then,[4]

$$d_m(p) = -\frac{s_{12}^{(m)}(p)}{s_{21*}^{(m)}(p)} = \frac{s_m(p)}{s_{22*}^{(m)}(p)} \quad , \tag{12}$$

$$s_m(p) = \frac{\Delta_m'(p)}{\Delta_{m-1}(p)} \cdot \prod_{r=0}^{m-1} d_r^{-1}(p) \quad , \tag{13}$$

$$1 - s_m(p)s_{m*}(p) = \frac{\Delta_m(p)\Delta_{m-2}(p)}{\Delta_{m-1}^2(p)} \quad , \tag{14}$$

$$m = 1 \rightarrow n; \quad d_0(p) = \Delta_{-1}(p) \equiv 1 \quad . \tag{15}$$

To help the reader develop an appreciation for the key diagnostic formula (13), we shall use it to solve a problem of considerable interest in its own right.

Problem 1. Determine the dielectric constants of a cascade of n+1 nonmagnetic ideal TEM lossless layers of the same 1-way delay τ by means of input-side time-domain reflectometry.

Solution. Clearly, layer m acts as a TEM line of characteristic impedance

$$R_m = \frac{377}{\sqrt{\epsilon_m}} \text{ ohms} \quad , \tag{16}$$

where $\epsilon_m > 0$ is its relative dielectric permittivity, $m = 0 \rightarrow n$. Further, all LLN' in Fig. ' collapse into a pair of perfectly conducting wires so that

[4] $s_{21}^{(m)}(p)$ and $s_{22}^{(m)}(p)$ cannot both be identically zero since $S^{(m)}(p)$ is regular paraunitary.

$$S^{(m)}(p) = \frac{1}{R_m + R_{m-1}} \cdot \left[\begin{array}{c|c} R_m - R_{m-1} & 2\sqrt{R_m R_{m-1}} \\ \hline 2\sqrt{R_m R_{m-1}} & R_{m-1} - R_m \end{array}\right] \qquad (17)$$

and $d_m(p) = -1$, $m = 1 \rightarrow n$. Consequently, the c_m's are real constants and

$$\frac{h^{(n)}(t)}{R_0} = \delta(t) + \sum_{m=1}^{n} 2c_m\, \delta(t-2m\tau) \quad . \qquad (18)$$

Of course, R_0 is obtained as the strength of the initial voltage impulse which appears at the input interminals.

Hence, in this all-transmission line case, $z_m(p) = R_m$,

$$s_m = \frac{R_m - R_{m-1}}{R_m + R_{m-1}} \qquad (19)$$

and

$$R_m = R_{m-1} \cdot \frac{1+s_m}{1-s_m} \quad , \quad m = 1 \rightarrow n \quad . \qquad (20)$$

According to (18), $h^{(n)}(t)$ is the sum of an initial impulse of strength R_0 and n successive reflected impulses of strengths $2R_0c_m$[5] separated by 2τ secs. Thus, all c_m's can be measured and (13) yields the explicit answer,

$$s_m = (-1)^{m-1} \cdot \frac{\Delta_m'}{\Delta_{m-1}} \quad , \quad m = 1 \rightarrow n \quad , \qquad (21)$$

where

$$\Delta_m = \det \begin{bmatrix} 1 & c_1 & c_2 & \cdot & \cdot & \cdot & c_m \\ c_1 & 1 & c_1 & \cdot & \cdot & \cdot & c_{m-1} \\ c_2 & c_1 & 1 & \cdot & \cdot & \cdot & c_{m-2} \\ \cdot & \cdot & \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & & \cdot \\ c_m & c_{m-1} & c_{m-2} & \cdot & \cdot & & 1 \end{bmatrix} \qquad (22)$$

and Δ_m' is the minor formed by striking out its first row and last column. Formulas (20) and (21) constitute the desired solution.

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[5]For $t > 0$ the input is open-circuited and this accounts for the doubling.

Before proceeding to the continuous taper it is necessary to clarify two conceptual points of great importance.

Point 1. First, we must show that the coefficients $c_m(p)$, $m=1\to n$, are actually determined uniquely by the segment of $h^{(n)}(t)$ over $0\leq t<2(n+1)\tau$. To this end, observe from (7) that

$$\frac{h^{(n)}(t)}{R_0} = 2h_1(t-2\tau)\ , \quad 2\tau\leq t<4\tau \quad , \tag{23}$$

and $h_1(t)$ is therefore known for $0\leq t<2\tau$. Since $h_1(t)$ is expressible as the sum of an impulse at $t=0$ and a finite number of polynomials in t multiplied by exponentials, its values over $2\tau\leq t<4\tau$ fix it completely for all $t\geq 0$ by analytic continuation.

Similarly, for $4\tau\leq t<6\tau$,

$$2h_2(t-4\tau) = \frac{h^{(n)}(t)}{R_0} - 2h_1(t-2\tau) \tag{24}$$

which means that $h_2(t)$ is also known for $0\leq t<2\tau$. Hence, by the above reasoning, it too is defined uniquely for all $t\geq 0$, etcetera.

Point 2. Second, although the coefficients $c_m(p)$, $m=1\to n$, are unique constructs from the given data $h^{(n)}(t)$, $0\leq t<2(n+1)\tau$, the application of formula (13) also requires knowledge of the n-1 functions $d_1(p), d_2(p),\ldots,d_{n-1}(p)$. In a great number of interesting situations it is possible to ascertain the d's in terms of R_0 and the c's with the aid of the following simple rule.

Rule (Appendix). Suppose that each LLN is all-pass free on its output side and D.C. transparent.[6] Let the strict Hurwitz polynomial $\alpha_k(p)$ be constructed with all the zeros of $\Delta_{k-1}(p)/\Delta_k(p)$ in $\mathrm{Re}\,p\leq 0$, multiplicities included. Then,

$$\prod_{r=0}^{k} d_r^{-1}(p) = (-1)^k\cdot\frac{\alpha_k(p)}{\alpha_{k*}(p)} \quad , \tag{25}$$

$$k=0\to n-1;\ \alpha_0(p)\equiv 1 \quad .$$

For example, if k=1, the polynomial $\alpha_1(p)$ is constructed with the closed left-half-plane zeros of

[6] LLN m is D.C. transparent if $S^{(m)}(0)$ is given by (17).

$$\frac{\Delta_0(p)}{\Delta_1(p)} = \frac{1}{1-c_1(p)c_{1*}(p)} \tag{26}$$

and

$$d_1^{-1}(p) = -\frac{\alpha_1(p)}{\alpha_{1*}(p)} \text{ , etc.} \tag{27}$$

Fortunately, physical tapers are usually constructed with material that cannot produce zeros of transmission in $\operatorname{Re} p > 0$ and their all-pass free character is intrinsic.

IV. DIAGNOSTICS FOR THE CONTINUOUS TAPER

We lay down the premise that the normalized scattering matrix $S^{(m)}(p)$ of a section of continuous taper lying between $x = m\tau$ and $x = (m+1)\tau$ approximates the identity transformation as $\tau \to 0$. That is, for every fixed p in Re p > 0,

$$\lim_{\tau \downarrow 0} S^{(m)}(p) = \left[\begin{array}{c|c} 0 & 1 \\ \hline 1 & 0 \end{array}\right] \quad . \tag{28}$$

Hence, to first order in τ and for Re p > 0,

$$\Lambda S^{(m)}(p) = 1_2 - \tau Z(x,p) \tag{29}$$

where

$$\Lambda = \left[\begin{array}{c|c} 0 & 1 \\ \hline 1 & 0 \end{array}\right] \tag{30}$$

and

$$Z(x,p) = \left[\begin{array}{c|c} z_{11}(x,p) & z_{12}(x,p) \\ \hline z_{21}(x,p) & z_{22}(x,p) \end{array}\right] \tag{31}$$

is a real rational matrix in p for every fixed x in $0 \leq x < T$.[7]

Since the taper is passive,

$$1_2 - S^{(m)*}(p)S^{(m)}(p) = \tau(Z(x,p) + Z^*(x,p)) \geq 0_2 \tag{32}$$

for all p in Re p > 0. It follows that

$$Z(x,p) + Z^*(x,p) \geq 0_2 \quad , \tag{33}$$

$$0 \leq x < T \; ; \; \text{Re}\, p > 0 \quad .$$

Again, because the taper is also lossless,

$$1_2 - S_*^{(m)}(p)S^{(m)}(p) = \tau(Z(x,p) + Z_*(x,p)) \equiv 0_2 \tag{34}$$

and Z(x, p) is forced to be skew-parahermitean in p:

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[7] The 2x2 matrix Z(x, p) supplies the taper's per-unit length description.

$$Z(x,p) + Z_*(x,p) \equiv 0_2 \quad , \tag{35}$$

$$0 \leq x < T; \text{ all } p \quad .$$

In short, for the class of passive lossless continuous tapers under consideration, the per-unit 2x2 descriptor $Z(x,p)$ is a real rational Foster matrix in p for every fixed x. In particular, $z_{11}(x,p)$ and $z_{22}(x,p)$ are reactances and

$$z_{21}(x,p) = -z_{12*}(x,p) \quad . \tag{36}$$

From Eqs. (29) and (31),

$$S^{(m)}(p) = \left[\begin{array}{c|c} -\tau z_{21}(x,p) & 1-\tau z_{22}(x,p) \\ \hline 1-\tau z_{11}(x,p) & -\tau z_{12}(x,p) \end{array}\right] \quad . \tag{37}$$

Hence, a comparison with (11) yields the first-order identifications,

$$s_m(p) = -\tau z_{21}(x,p) \quad , \tag{38}$$

$$s_{12}^{(m)}(p) = 1 - \tau z_{22}(x,p) = \exp\left[-\tau z_{22}(x,p)\right] \quad , \tag{39}$$

$$s_{21}^{(m)}(p) = 1 - \tau z_{11}(x,p) = \exp\left[-\tau z_{11}(x,p)\right] \quad , \tag{40}$$

$$s_{22}^{(m)}(p) = -\tau z_{11}(x,p) = -s_{m*}(p) \tag{41}$$

and

$$d_m(p) = -\frac{s_{12}^{(m)}(p)}{s_{21*}^{(m)}(p)} = -\exp\left[-2\tau\xi(x,p)\right] \tag{42}$$

where[8]

$$\xi(x,p) = \frac{z_{11}(x,p) + z_{22}(x,p)}{2} \quad . \tag{43}$$

Clearly, $\xi(x,p)$ is a real rational reactance function of p for every fixed x. Moreover, as is readily appreciated from (42),

---

[8] If the taper is reciprocal, $s_{12}^{(m)}(p) = s_{21}^{(m)}(p)$ so that $z_{11}(x,p) = z_{22}(x,p)$ and $\xi(x,p) = z_{11}(x,p)$.

$$\tau_g(x,\omega) \equiv \left.\frac{d\xi(x,p)}{dp}\right|_{p=j\omega} \tag{44}$$

represents the average to-and-fro group delay per second of taper at location x due to dispersion.[9]

We can now recognize with the aid of (42) that for $\tau \downarrow 0$ and $x = m\tau$ fixed,

$$\prod_{r=0}^{m-1} d_r^{-1}(p) = (-1)^{m-1} \cdot \exp\left[2\tau \sum_{r=0}^{m-1} \xi(r\tau, p)\right] + O(\tau) \tag{45}$$

$$= (-1)^{m-1} \cdot \exp\left[2 \int_0^x \xi(y,p)dy\right] + O(\tau) \quad . \tag{46}$$

Consequently, since $s_m(p)/\tau = -z_{21}(x,p)$, Eqs. (15), (13) and (46) yield the fundamental <u>continuous-discrete</u> inversion formula,

$$P(x) + \eta(x,p) = -z_{21}(x,p) = \exp\left[2\int_0^x \xi(y,p)dy\right] \cdot \lim_{\tau \downarrow 0} (-1)^{m-1} \cdot \frac{\Delta_m'(p)}{\tau \Delta_{m-1}(p)} ,$$

$$x = m\tau \text{ fixed}; \quad 0 \leq x < T \quad . \tag{47}$$

In general, $z_{21}(x,p)$ determines only the sum $P(x)+\eta(x,p)$ and not $P(x)$ and $\eta(x,p)$ individually. However, if the taper is D. C. transparent so that it reduces to a pair of perfectly conducting wires at $\omega=0$, then $\eta(x,0) = \xi(x,0) = 0$ and

$$P(x) = -z_{21}(x,0) \quad . \tag{48}$$

Thus,

$$\eta(x,p) = z_{21}(x,0) - z_{21}(x,p) \tag{49}$$

and the taper profile is completely specified by $z_{21}(x,p)$.

---

[9] Let $\xi(x,j\omega) = j\theta(x,\omega)$, ω real. Then, from the positive-slope property of reactances,

$$\tau_g(x,\omega) = \frac{d\theta(x,\omega)}{d\omega} \geq 0 \quad , \tag{49a}$$

all ω.

To exploit the basic inversion formula (47), it is at least necessary to exhibit a first-order relationship between the general coefficient $c_m(p)$ and the segment of the impulse response h(t) of the continuous taper over the time interval $0 \leq t < 2T$. From the affirmative answer to the question posed in point 1 it is reasonable to infer that such an identification exists and it is expected that the principle developed in the next section will play an important role.

V. THE WEAK CONVERGENCE PRINCIPLE

The structure of $h^{(n)}(t)/R_o$, the normalized impulse response of the $n^{\underline{th}}$ order discrete-cascade approximation to the continuous taper, is shown in Eq. (7). From this expression it appears intuitively evident that the normalized impulse response $h(t)/R_0$ of the continuous taper must have the form

$$\frac{h(t)}{R_0} = \delta(t) + 2u(t) \quad , \tag{50}$$

where $u(t)$ is continuous for $t > 0$ and assumes the true initial value $u(0) = 0$.[10]

Let $i_1(t)$ and $i_2(t)$ be any two smooth input current drivers[11] and let $v_1(t)$, $v_1^{(n)}(t)$ and $v_2(t)$, $v_2^{(n)}(t)$ denote the respective zero-state input voltage responses at the terminals of the continuous taper and its discrete $n^{\underline{th}}$ order approximation. We make the consistent physical assumption that the energy supplied to the approximation up to any time t converges as $n \to \infty$ to that supplied to the continuous taper up to time t. Symbolically,

$$\lim_{n \to \infty} \int_o^t v_k^{(n)}(y) i_k(y) dy = \int_o^t v_k(y) i_k(y) dy \quad , \tag{51}$$

$$0 \le t < 2T \; ; \; k = 1, 2 \quad .$$

Since the taper and the discrete cascade are linear networks, the current $i_1(t) + i_2(t)$ produces the respective input voltages $v_1(t) + v_2(t)$ and $v_1^{(n)}(t) + v_2^{(n)}(t)$. Thus, (51) also implies that

$$\lim_{n \to \infty} \int_o^t (v_1^{(n)} + v_2^{(n)})(i_1 + i_2) dy = \int_o^t (v_1 + v_2)(i_1 + i_2) dy \tag{52}$$

and by subtraction we obtain

$$\lim_{n \to \infty} \int_o^t (v_1^{(n)} i_2 + v_2^{(n)} i_1) dy = \int_o^t (v_1 i_2 + v_2 i_1) dy \quad , \tag{53}$$

$$0 \le t < 2T \quad .$$

Choose $i_1(t) = \delta(t)$. Then, by definition, $v_1^{(n)}(t) = h^{(n)}(t)$ and $v_1(t) = h(t)$ so that (53) yields

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[10] In general, $u(0^+) \neq 0$.

[11] $i_1(t)$ and $i_2(t)$ are infinitely differentiable for $t \geq 0$ and all their derivatives vanish for $t = 0$.

$$v_2^{(n)}(0) + \underset{n\to\infty}{\text{limit}} \int_0^t h^{(n)}(y) i_2(y) dy = v_2(0) + \int_0^t h(y) i_2(y) dy \ , \qquad (54)$$

$$0 \leq t < 2T \ .$$

Hence, for <u>every</u> choice of smooth driver i(t),

$$\underset{n\to\infty}{\text{limit}} \sum_{m=1}^{n} \int_0^t h_m(y-2m\tau) i(y) dy = \int_0^t u(y) i(y) dy, \ \ 0 \leq t < 2T \ , \qquad (55)$$

and

$$\underset{n\to\infty}{\text{limit}} \ v^{(n)}(0) = v(0) \ . \qquad (56)$$

In words, the impulse response of the discrete cascade converges <u>weakly</u> (in the sum of functional analysis in Hilbert space) to the impulse response of the continuous taper:

$$h^{(n)}(t) \rightharpoonup h(t) \ , \ \ 0 \leq t < 2T \ . \qquad (57)$$

We are now in a position to derive a time-domain integral equation which solves the inverse problem for the dispersionless taper ($\xi \equiv 0$). This taper is usually described by the telegrapher's equation.

VI. INVERSION OF THE CLASSICAL TELEGRAPH EQUATION

By definition, the taper is without dispersion if

$$\tau_g(x,\omega) = \frac{d\theta(x,\omega)}{d\omega} = \frac{d\xi(x,p)}{dp}\bigg|_{p=j\omega} \equiv 0 \tag{58}$$

in ω for every fixed x; i.e., if $\xi(x,p)$ is a constant in p. However, since $\xi(x,p)$ is a reactance this constant must equal zero whence,

$$z_{11}(x,p) + z_{22}(x,p) \equiv 0 \ . \tag{59}$$

It now follows easily from the Foster character of $Z(x,p)$ that

$$z_{11}(x,p) = z_{22}(x,p) \equiv 0 \tag{60}$$

and that $z_{21}(x,p)$ = a frequency-independent constant. But then, if the taper is also D.C. transparent, $\eta(x,0) = 0$ and we conclude from (48) that this constant equals $-P(x)$. To summarize, for a D.C. transparent dispersionless taper,

$$P(x) = \underset{\tau\downarrow 0}{\text{limit}} (-1)^{m-1} \cdot \frac{\Delta_m'(p)}{\tau\Delta_{m-1}(p)}, x = m\tau . \tag{61}$$

Thus, in the absence of dispersion the taper profile is delineated solely by its logarithmic taper

$$P(x) = \frac{1}{2}\,\frac{d\ln R(x)}{dx} \tag{62}$$

and the discrete approximation is nothing more than a cascade of n+1 ideal TEM lines of the same 1-way delay τ.

Consequently, invoking the results in Problem 1, we find that

$$h_m(t) \Leftrightarrow c_m\delta(t) \tag{63}$$

where all the coefficients c_m are real constants, $m = 1 \to n$. Concomitantly, Δ_m' and Δ_{m-1} are independent of p and for every $n \geq 0$,

$$\frac{h^n(t)}{R_0} = \delta(t) + \sum_{m=1}^{n} 2c_m\delta(t-2m\tau) \ , \tag{64}$$

$$0 \leq t < 2(n+1)\tau \ .$$

Select any fixed t in $0 \le t < 2T$ and choose the integer k so that

$$2k\tau \le t < 2(k+1)\tau \qquad (65)$$

where

$$\tau = \frac{T}{n+1} \quad . \qquad (66)$$

According to (55), for any smooth i(t),

$$\lim_{n\to\infty} \sum_{m=1}^{k} c_m i(2m\tau) = \int_0^t u(y)i(y)dy \quad . \qquad (67)$$

But from the definition of a Riemann integral,

$$\int_0^t u(y)i(y)dy = \lim_{n\to\infty} \sum_{m=1}^{k} 2\tau u(2m\tau)i(2m\tau) \qquad (68)$$

and a comparison with (67) immediately leads to the first-order connection formula,

$$c_m = 2\tau u(2m\tau) \quad , \quad m = 1 \to n \quad . \qquad (69)$$

As expected, as $\tau \to 0$, $h^{(n)}(t)$ provides an increasingly accurate sampled-data version of h(t).

Theorem 2. Let

$$\frac{h(t)}{R_o} = \delta(t) + 2u(t) \qquad (70)$$

constitute a (normalized) record of the *impulse response of a dispersionless* D.C. transparent continuous taper over the time interval $0 \le t < 2T$. For fixed x, $0 < x < T$, let $\psi(t)$ denote the solution of the integral equation

$$\psi(t) + \int_0^{2x} u(|t-y|)\psi(y)dy = \delta(t) \quad , \qquad (71)$$

$$0 \le t < 2x \quad .$$

Then,

$$P(x) = -2\psi(2x), \quad 0 < x < T \quad . \qquad (72)$$

Proof. (Largely heuristic). Let

$$\psi(t) = \delta(t) + \varphi(t) \quad . \tag{73}$$

Clearly, $\psi(t) = \varphi(t)$, $t > 0$, and

$$\varphi(t) + \int_0^{2x} u(|t-y|)\, \varphi(y)dy = -u(t) \quad , \tag{74}$$

$$0 \leq t < 2x \quad .$$

This conventional distribution-free Fredholm integral equation of the second kind is the precise equivalent of (71).

With τ given by (66) and the integer m chosen so that $m\tau < x < (m+1)\tau$, we set $\varphi(2k\tau) = x_k$, $k = 0 \to m$, and evaluate the integral as a Riemann sum at the points $t = 0, 2\tau, .., 2m\tau$ with the aid of (69). The result is the linear system,[12]

$$\begin{bmatrix} 1 & c_1 & c_2 & \cdot & \cdot & \cdot & c_m \\ c_1 & 1 & c_1 & \cdot & \cdot & \cdot & c_{m-1} \\ c_2 & c_1 & 1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ c_m & c_{m-1} & c_{m-2} & \cdot & \cdot & \cdot & 1 \end{bmatrix} \begin{bmatrix} x_o + \frac{1}{2\tau} \\ x_1 \\ x_2 \\ \cdot \\ \cdot \\ x_m \end{bmatrix} = \begin{bmatrix} \frac{1}{2\tau} \\ 0 \\ 0 \\ \cdot \\ \cdot \\ 0 \end{bmatrix} \quad . \tag{75}$$

(Observe that the derivation of (75) has made use of the initial value constraint, $u(0) = 0$.)

It now follows easily from Cramer's rule that

$$\frac{\Delta_{m-1}}{\Delta_m} = 2\tau x_o + 1 \tag{76}$$

and

$$(-1)^m \cdot \frac{\Delta_m'}{\tau \Delta_m} = 2x_m \quad . \tag{77}$$

Thus,

[12] This algebraic procedure works in a general L_2-setting [4].

$$-\frac{2x_m}{2\tau x_o+1} = (-1)^{m-1} \cdot \frac{\Delta_m'}{\tau\Delta_{m-1}} \tag{78}$$

and

$$-2 \lim_{\tau \downarrow 0} x_m = -2\varphi(2x) = -2\psi(2x) = \lim_{\tau \downarrow 0} (-1)^{m-1} \cdot \frac{\Delta_m'}{\tau\Delta_{m-1}} = P(x) \quad , \tag{79}$$

Q.E.D.

Comment 1. The determination of $P(x)$ for $0 < x < T$ has been reduced to the solution of a 1-parameter family of Fredholm integral equations of the second kind (with parameter x). Moreover, because of the symmetric positive-definite character of the coefficient matrix in (75) and the sampled-data character of the c_m's, equation (79) can be made the basis of a stable digital algorithm of the Levinson type which operates in real time. In this setting it is of course natural and (advisable) to exploit the theory of orthogonal polynomials on the unit circle.

As an illustrative example, consider the exponential line whose Laplace transform description is given by

$$\frac{dV(x,p)}{dx} = -pR(x)I(x,p) \quad , \tag{80}$$

$$\frac{dI(x,p)}{dx} = -pR^{-1}(x)V(x,p) \tag{81}$$

where[13]

$$R(x) = R_0 e^{2ax} \, , \quad 0 \leq x < T \quad . \tag{82}$$

As usual, R_0 and a are real constants, the former being positive, and

$$v(x,t) \Leftrightarrow V(x,p) \quad , \tag{83}$$

$$i(x,t) \Leftrightarrow I(x,p) \quad . \tag{84}$$

Thus,

$$P(x) = \frac{1}{2} \cdot \frac{d\ln R(x)}{dx} = a \quad , \tag{85}$$

[13]Note that x is in secs. since the local velocity of propgation has already been normalized to unity.

a constant.

To calculate the corresponding input voltage impulse response $v(0,t) = h(t)$, we first eliminate I to obtain the second-order differential equation for V,

$$\frac{d^2V}{dx^2} - 2a\frac{dV}{dx} - p^2 V = 0 \quad . \tag{86}$$

Hence,

$$V(x,p) = Ae^{(a-\delta)x} + Be^{(a+\delta)x} \tag{87}$$

where A(p) and B(p) are independent of y and

$$\delta(p) = \sqrt{p^2 + a^2} \quad . \tag{88}$$

(We choose that branch of the square-root for which $\delta(p) > 0$ for $p > 0$. This guarantees that $\delta(p)$ is positive-real in $\mathrm{Re}\, p > 0$.)

If the line is semi-infinite, $B(p) = 0$[14] and we find that for $\mathrm{Re}\, p > 0$

$$V(x,p) = Ae^{(a-\delta)x} \quad , \tag{89}$$

$$I(x,p) = \frac{(\delta - a)A}{pR(x)} e^{(a-\delta)x} \tag{90}$$

and

$$Z_{in}(p) \equiv \frac{V(0,p)}{I(0,p)} = \frac{pR_0}{\sqrt{p^2+a^2} - a} \quad .$$

Thus, since $h(t) \Leftrightarrow Z_{in}(p)$, (91) yields

$$\frac{h(t)}{R_0} \Leftrightarrow \frac{\sqrt{p^2+a^2} - a}{p} \quad , \quad \mathrm{Re}\, p > 0 \quad , \tag{92}$$

and carrying out the inversion we obtain the explicit result

$$\frac{h(t)}{R_0} = \delta(t) + 2u(t) \quad , \tag{93}$$

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[14] The exponential $e^{(a+\delta)x} \to \infty$ as $x \to \infty$ since $a+\delta(p)$ has positive real-part in the right-half p-plane.

where[15]

$$u(t) = \frac{a}{2}\left(1 + \int_o^{at} J_o(y)dy - J_1(at)\right) \tag{94}$$

$$= a\left(\frac{1 + J_1(at)}{2} + \sum_{r=1}^{\infty} J_{2r+1}(at)\right), \tag{95}$$

$$0 \leq t < 2T \quad .$$

Note that the analytic form of the impulse response is quite complicated, even for this very simple nonuniform line.

Another excellent and challenging example is the trigonometric line for which

$$R(x) = R_0/\cos^2 x \; , \quad 0 \leq x < T \quad . \tag{96}$$

Clearly,

$$P(x) = \tan x \tag{97}$$

and the behavior close to $x = \pi/2$ is most severe.

The relevant Diff. Eq. for $V(x, p)$ is easily found to be

$$\frac{d^2V}{dx^2} - 2\tan x \cdot \frac{dV}{dx} - p^2 V = 0 \tag{98}$$

and its solution is given by

$$V(x, p) = (A(p)\cosh\sqrt{p^2-1}\, y + B(p)\sinh\sqrt{p^2-1}\, y)/\cos x, \; \mathrm{Re}\, p > 0 \; . \tag{99}$$

It is now a straightforward matter to derive the relatively simple expression

$$\frac{h(t)}{R_0} = \delta(t) + I_1(t), \quad t \geq 0 \quad , \tag{100}$$

where $I_1(t)$ is the modified Bessel function of order one. Clearly,

$$u(t) = I_1(t)/2 \quad . \tag{101}$$

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[15] $J_r(y)$ is the r^{th} order Bessel function of the first kind.

(The agreement between the exact solutions (85) and (97) and those obtained by the use of (72) has been encouraging but a further refinement of the software is still needed.)

In Part 2 of this report we hope to be able to describe the complete solution for the dispersive case. Our research up to this point appears to indicate that the unknown per-unit length group-delay function $\xi(x, p)$ introduces an additional integral equation. The final formulation will probably involve a 1-parameter family of two coupled Fredholm integral equations of the second kind.

APPENDIX 1

Let the reactance 2-port N' delimited between the dotted lines in Fig. 1b be described by its 2x2 scattering matrix $S(p)$ normalized to R_0 on the left and R_n on the right. It can be shown by straightforward analysis [3] that there exist two real 2-variable polynomials

$$h(p, z) = \sum_{r=0}^{n} z^r h_r(p) \tag{1.1}$$

and

$$g(p, z) = \sum_{r=0}^{n} z^r g_r(p) \tag{1.2}$$

in p and z and a real polynomial $f(p) \not\equiv 0$ in the single variable p such that[16]

$$S(p) = W(p, e^{-2p\tau}) = W(p, z)\Big|_{z=e^{-2p\tau}} \tag{1.3}$$

where

$$W(p, z) = \frac{1}{g(p, z)}\left[\begin{array}{c|c} h(p, z) & e^{-np\tau} f(p) \\ \hline e^{-np\tau} f_*(p) & -z^n h_*(p, z) \end{array}\right]. \tag{1.4}$$

The matrix $W(p, z)$ is 2-variable bounded-real (b.r.), $g(p, z)$ is 2-variable Hurwitz and $g_0(p)$ is Hurwitz; i.e., if we let

$$D^+(1) = \{(p, z): \operatorname{Re} p > 0,\ |z| < 1\}, \tag{1.5}$$

it is true that

$$1_2 - W^*(p, z)\, W(p, z) \geq 0_2 \tag{1.6}$$

and

$$g(p, z) \neq 0 \tag{1.7}$$

for all pairs $(p, z) \in D^+(1)$ and

$$g_0(p) \neq 0, \quad \operatorname{Re} p > 0 . \tag{1.8}$$

[16] The use of z as a symbol for impedance and as a complex variable should cause no confusion.

In addition, $h(p,z)$, $g(p,z)$ and $f(p)$, viewed as polynomials in two abstract variable p and z, are tied together by the Feldtkeller relation

$$g(p,z)g_*(p,z) - h(p,z)h_*(p,z) = f(p)f_*(p) \equiv \nu(p) \not\equiv 0 \quad . \tag{1.9}$$

Our immediate aim is to exploit the evident intimate connection between the 2-variable input reflection coefficient

$$s_{11}(p,z) = \frac{h(p,z)}{g(p,z)} \tag{1.10}$$

and the backend input reflection *coefficient*

$$s_{22}(p,z) = -\frac{z^n h_*(p,z)}{g(p,z)} \quad . \tag{1.11}$$

This we accomplish by means of the following piece of physical *insight*.

Let $Z_1(p,z)$ and $Z_2(p,z)$ denote the 2-variable driving-point impedances seen looking into the front and back ends of N' with the opposite ports closed on R_n and R_0, respectively. Since z is ultimately to be identified with $e^{-2p\tau}$ which approaches zero as $\tau \to \infty$ for all fixed p in Re $p > 0$, it is obvious that $Z_2(p,0)$ is nothing more than the backend input impedance of N' with all lines infinitely long. However, since the input impedance of an infinitely long line equals its characteristic impedance, this means that $Z_2(p,0)$ is the impedance seen looking into the backend of LLN n with its input port closed on R_{n-1}. Thus, in the notation of (11),

$$s_{22}(p,0) = \frac{Z_2(p,0) - R_n}{Z_2(p,0) + R_n} = s_{22}^{(n)}(p) \quad . \tag{1.12}$$

Consequently, using (1.1), (1.2), and (1.11) we obtain,

$$s_{22}(p,z) = -\frac{h_{n*}(p) + zh_{n-1*}(p) + \cdots + z^n h_{0*}(p)}{g_0(p) + zg_1(p) + \cdots + z^n g_n(p)} \tag{1.13}$$

and

$$s_{22}(p,0) = s_{22}^{(n)}(p) = -\frac{h_{n*}(p)}{g_0(p)} \quad , \tag{1.14}$$

a key result.

By definition,

$$Z_1 = R_0 \cdot \frac{1+s_{11}}{1-s_{11}} = R_0 \cdot \frac{g+h}{g-h} \tag{1.15}$$

and a direct calculation yields

$$\frac{Z_1+Z_{1*}}{2R_0} = \frac{gg_* - hh_*}{(g-h)(g-h)_*} = \frac{\nu}{(g-h)(g-h)_*} = \frac{\nu}{ee_*} \quad , \tag{1.16}$$

where

$$e(p,z) \equiv g(p,z) - h(p,z) = \sum_{r=0}^{n} z^r (g_r - h_r) = \sum_{r=0}^{n} z^r e_r(p) \quad . \tag{1.16a}$$

Clearly, $Z_1(p,0) = R_0$ so that

$$0 \equiv s_{11}(p,0) = \frac{h_0(p)}{g_0(p)} \quad . \tag{1.17}$$

Hence,

$$h_0(p) \equiv 0 \quad . \tag{1.18}$$

From (1.9),

$$g_n(p) g_{0*}(p) - h_n(p) h_{0*}(p) \equiv 0 \tag{1.19}$$

and (1.18) forces

$$g_n(p) \equiv 0 \tag{1.20}$$

since $g_0(p) \not\equiv 0$. Thus,

$$e_0(p) = g_0(p) - h_0(p) = g_0(p) \quad , \tag{1.21}$$

$$e_n(p) = g_n(p) - h_n(p) = -h_n(p) \tag{1.22}$$

and we conclude from (1.14) that

$$s_{22}^{(n)}(p) = \frac{e_{n*}(p)}{e_0(p)} \quad . \tag{1.23}$$

Since $Z_1(p,z)/R_0$ is 2-variable p.r. it admits an expansion

$$\frac{Z_1(p,z)}{R_0} = 1 + 2 \sum_{r=1}^{\infty} z^r c_r(p) \tag{1.24}$$

which converges for all z in $|z| \leq 1$ for every fixed p in $\mathrm{Re}\, p > 0$ and for all but a finite number of p's on the $j\omega$-axis. These assertions are easily validated by making use of the Schur criterion for functions with positive real-part in $|z| < 1$. Namely, for every $r \geq 1$,

$$Z^{(r)}(p) = \begin{bmatrix} 1 & & & & & \\ 2c_1(p) & 1 & & & & \\ 2c_2(p) & 2c_1(p) & 1 & & & \\ \cdot & \cdot & \cdot & \cdot & & \\ \cdot & \cdot & \cdot & & \cdot & \\ \cdot & \cdot & \cdot & & & \\ 2c_r(p) & 2c_{r-1}(p) & \cdot & \cdot\ \cdot & 2c_1(p) & 1 \end{bmatrix} \tag{1.25}$$

is a rational p.r. matrix [2].

According to (1.16),

$$\left(1 + \sum_{r=1}^{\infty} z^r c_r(p) + \sum_{r=1}^{\infty} z^{-r} c_{r*}(p)\right)\left(e_0(p) + ze_1(p) + \ldots + z^n e_n(p)\right) = \frac{\nu(p)}{e_*(p, z)} \tag{1.26}$$

and by equating coefficients of z^r on both sides, $r = 0 \to n$, we derive the linear matrix equation,

$$\begin{bmatrix} 1 & c_{1*}(p) & \cdot\ \cdot\ \cdot & c_{n*}(p) \\ c_1(p) & 1 & \cdot\ \cdot\ \cdot & c_{n-1*}(p) \\ \cdot & \cdot & \cdot\ \cdot\ \cdot & \cdot \\ \cdot & \cdot & \cdot\ \cdot\ \cdot & \cdot \\ c_n(p) & c_{n-1}(p) & \cdot\ \cdot\ \cdot & 1 \end{bmatrix} \underset{\sim}{e}(p) = \frac{\nu(p)}{e_{0*}(p)} \begin{bmatrix} 1 \\ 0 \\ \cdot \\ \cdot \\ 0 \end{bmatrix} \tag{1.27}$$

where

$$\underset{\sim}{e}(p) = \begin{bmatrix} e_0(p) \\ e_1(p) \\ \cdot \\ \cdot \\ e_n(p) \end{bmatrix}. \tag{1.28}$$

From Cramer's rule,

$$e_0(p) = \frac{\nu(p)}{e_{0*}(p)} \cdot \frac{\Delta_{n-1}(p)}{\Delta_n(p)} \tag{1.29}$$

and

$$e_n(p) = (-1)^n \cdot \frac{\nu(p)}{e_{0*}(p)} \cdot \frac{\Delta_{n'}(p)}{\Delta_n(p)} \quad . \tag{1.30}$$

Thus, invoking (1.23),

$$s_{22}^{(n)}(p) = \frac{e_{n*}(p)}{e_0(p)} = (-1)^n \cdot \frac{e_{0*}(p)}{e_0(p)} \cdot \frac{\Delta_{n'*}(p)}{\Delta_{n-1}(p)} \quad . \tag{1.31}$$

Now if we go back to Eq. (12) and substitute (1.31) we find the expression

$$s_n(p) = (-1)^n \cdot d_n(p) \cdot \frac{e_0(p)}{e_{0*}(p)} \cdot \frac{\Delta_{n'}(p)}{\Delta_{n-1}(p)} \tag{1.32}$$

for the junction reflection coefficient of LLN n.

In view of (1.29), $e_0(p)$ must be determined from the equation

$$e_0(p)e_{0*}(p) = \nu(p) \cdot \frac{\Delta_{n-1}(p)}{\Delta_n(p)} \quad . \tag{1.33}$$

Since $e_0(p) = g_0(p)$ is Hurwitz, it coincides with the Wiener factor of the right-hand side of (1.33). In general, therefore, $e_0(p)$ depends not only on the coefficients $c_1(p), c_2(p), \cdots, c_n(p)$ but also on that part of $\nu(p)$ not cancelled by the even (nonnegative) rational function $\Delta_{n-1}(p)/\Delta_n(p)$.

To examine the matter more closely, let

$$\frac{\Delta_{n-1}(p)}{\Delta_n(p)} = \frac{\alpha_n(p)\alpha_{n*}(p)}{\beta_n(p)\beta_{n*}(p)} \tag{1.34}$$

where $\alpha_n(p)$ and $\beta_n(p)$ are two relatively prime Hurwitz polynomials. Evidently, $\beta_n\beta_{n*}$ divides ν because the product e_0e_{0*} is polynomial. More strongly [3], if all LLN's are all-pass free on their output sides, $\nu(p) = \beta_n(p)\beta_{n*}(p)$ and

$$e_0(p) = \alpha_n(p) \quad ; \tag{1.35}$$

i.e., the Hurwitz polynomial $e_0(p)$ is formed with <u>all</u> the zeros of the quotient $\Delta_{n-1}(p)/\Delta_n(p)$ in $\mathrm{Re}\, p \leq 0$.

Although we omit the proof, it is easily shown by induction [3] that without restriction,

$$\frac{e_0(p)}{e_{0*}(p)} = (-1)^n \cdot \prod_{r=1}^{n} d_r^{-1}(p) \quad . \tag{1.36}$$

This granted, (1.32) then yields

$$s_n(p) = \frac{\Delta_{n'}(p)}{\Delta_{n-1}(p)} \cdot \prod_{r=1}^{n-1} d_r^{-1}(p) \quad . \tag{1.37}$$

Of course, the above derivation applies for all $m = 1 \to n$ and we recover the diagnostic formula (13) of theorem 1.

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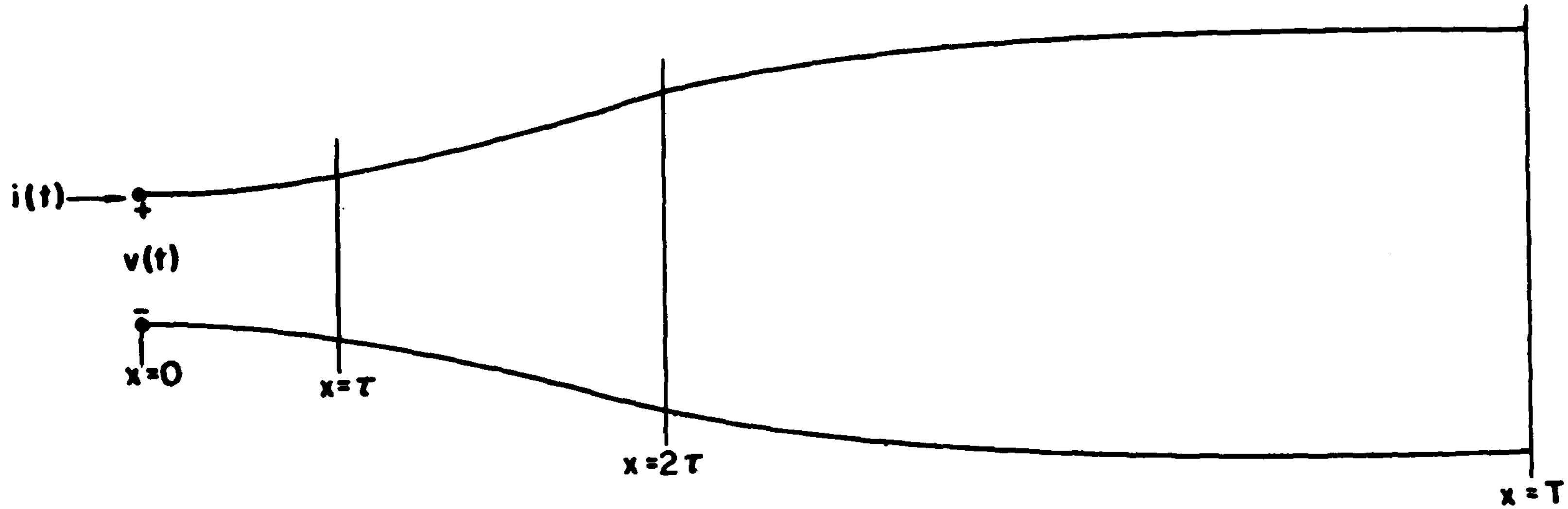
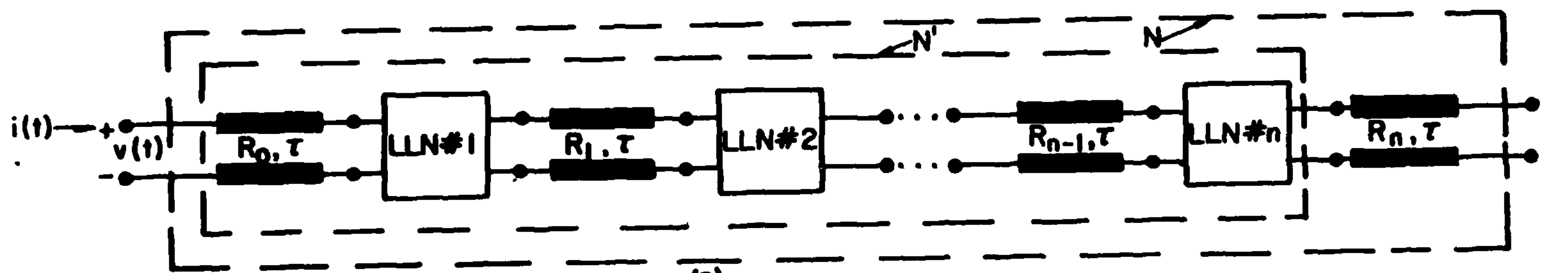


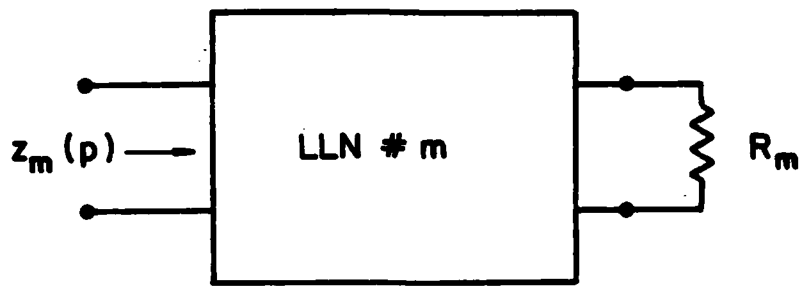
IF $i(t) = \delta(t)$, $v(t) = h(t)$; $0 \le t < 2T$

FIG. 1a THE CONTINUOUS TAPER: SMOOTH, PASSIVE AND LOSSLESS



IF $i(t) = \delta(t)$, $v(t) = h^{(n)}(t)$; $0 \le t < 2(n+1)\tau$

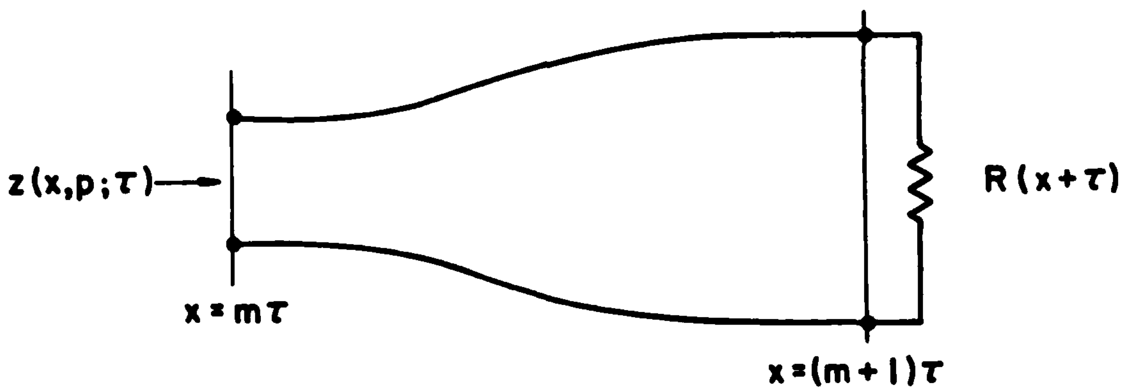
FIG. 1b A DISCRETE n^{th} ORDER APPROXIMATION TO THE CONTINUOUS TAPER: EACH LLN IS A LUMPED, PASSIVE LOSSLESS 2-PORT AND THE $n+1$ LINES ARE IDEAL TEM WITH THE SAME 1-WAY DELAY $\tau > 0$



$$s_m(p) = \frac{z_m(p) - R_{m-1}}{z_m(p) + R_{m-1}}$$

$$R_m > 0, \quad m = 0 \longrightarrow n \qquad \text{(a)}$$

$$R(x) > 0, \quad 0 \le x < T$$



(b)

$$s_m(p) + o(x,p;\tau) = \frac{z(x,p;\tau) - R(x)}{z(x,p;\tau) + R(x)}$$

FIG. 2a, 2b THE MEANING OF JUNCTION REFLECTION COEFFICIENT

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